The z-Transform

B Tech VIth Semester EIC- DSP

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INTRODUCTION

Why z-Transform?

A generalization of Fourier transform

Why generalize it?

- FT does not converge on all sequence
- Notation good for analysis
- Bring the power of complex variable theory deal with the discrete-time signals and systems

The z-Transform

Z-TRANSFORM

Definition

The *z*-transform of sequence *x*(*n*) is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- Let $z = e^{-j\omega}$.

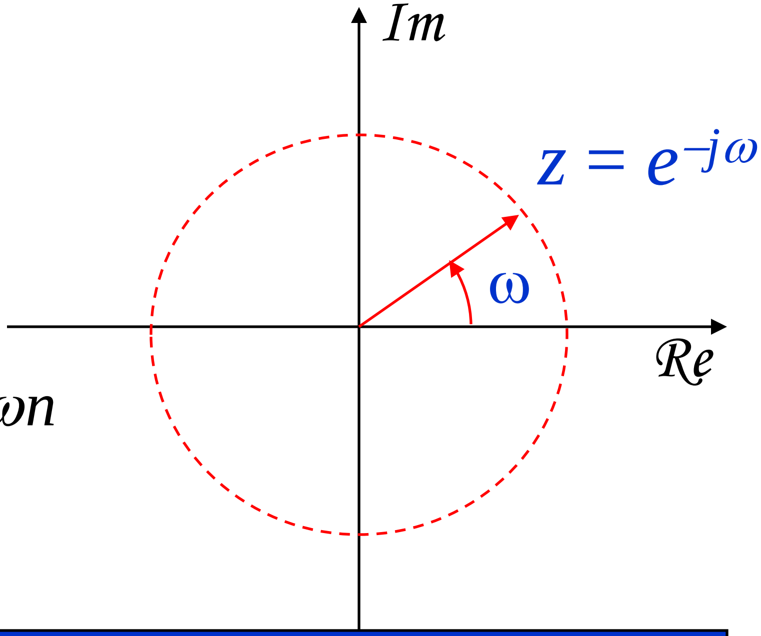
Fourier Transform

$$\Rightarrow \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

z-Plane

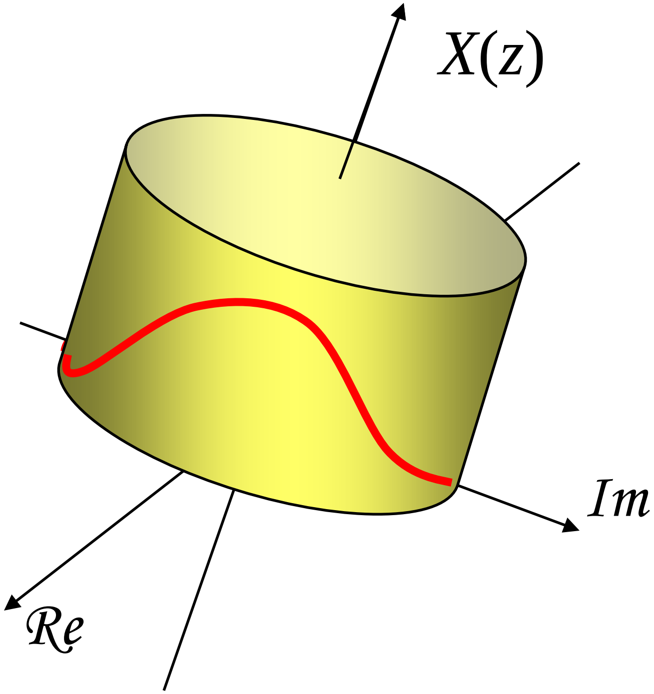
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

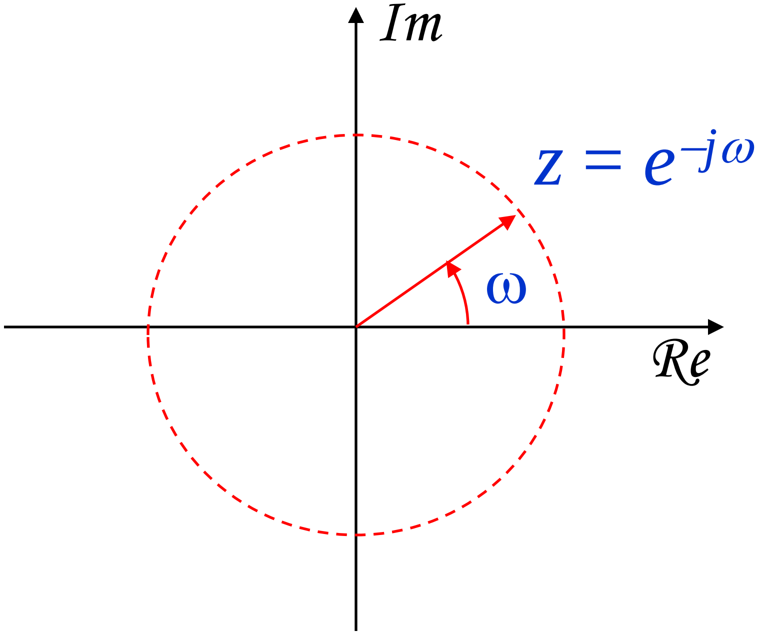
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$



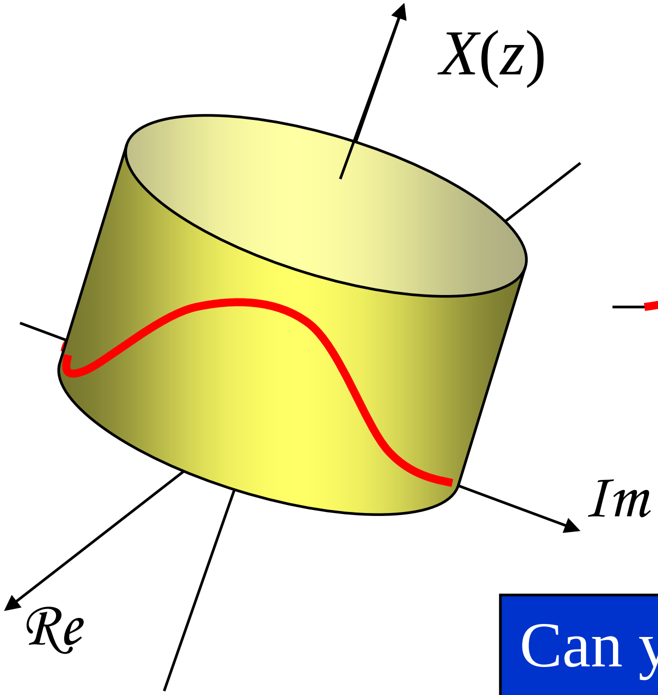
Fourier Transform is to *evaluate z-transform on a unit circle.*

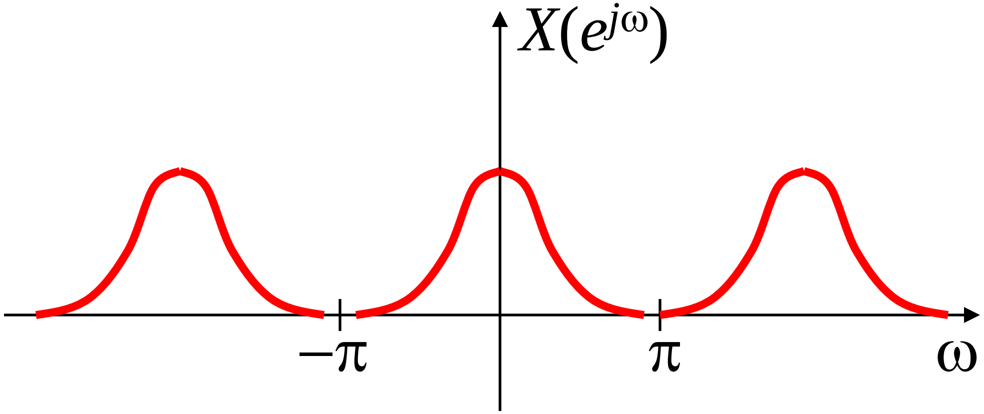
z-Plane





Periodic Property of FT





Can you say why Fourier Transform is a periodic function with period 2π?

The z-Transform

ZEROS AND POLES

Definition

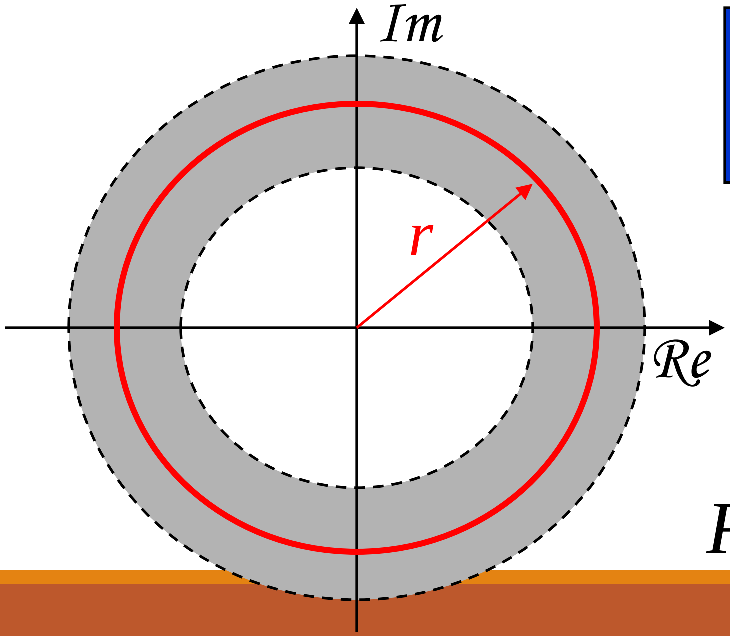
Give a sequence, the set of values of z for which the z-transform converges, i.e., $|X(z)|<\infty$, is called the region of convergence.

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) z^{-n} \right| = \sum_{n=-\infty}^{\infty} |x(n)| |z|^{-n} < \infty$$

ROC is centered on origin and consists of a set of rings.

Example: Region of Convergence

$$|X(z)| = \left|\sum_{n=-\infty}^{\infty} x(n) z^{-n}\right| = \sum_{n=-\infty}^{\infty} |x(n)||z|^{-n} < \infty$$



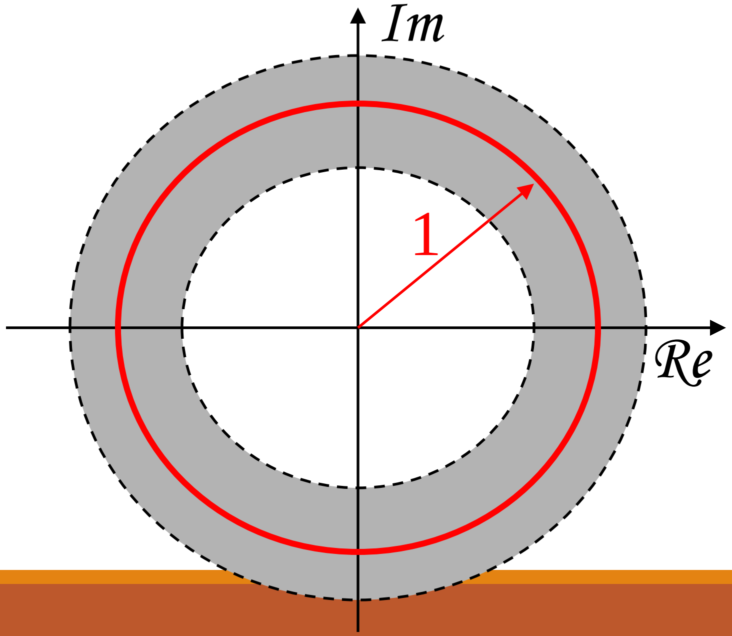
ROC is an annual ring centered on the origin.

$$R_{x-} < |z| < R_{x+}$$

$$ROC = \{z = re^{j\omega} \mid R_{x-} < r < R_{x+}\}$$

Stable Systems

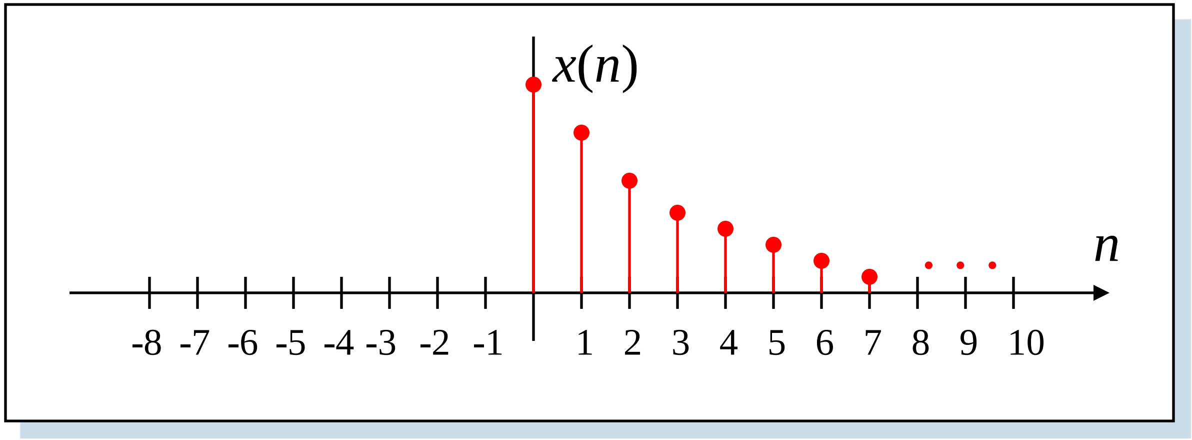
A stable system requires that its Fourier transform is uniformly convergent.



- Fact: Fourier transform is to evaluate *z*-transform on a unit circle.
- A stable system requires the ROC of *z*-transform to include the unit circle.

Example: A right sided Sequence

$$x(n) = a^n u(n)$$



Example: A right sided Sequence

$$\boxed{x(n) = a^n u(n)}$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |az^{-1}| < \infty \Rightarrow |az^{-1}| < 1$$

$$\Rightarrow |z| > |a|$$

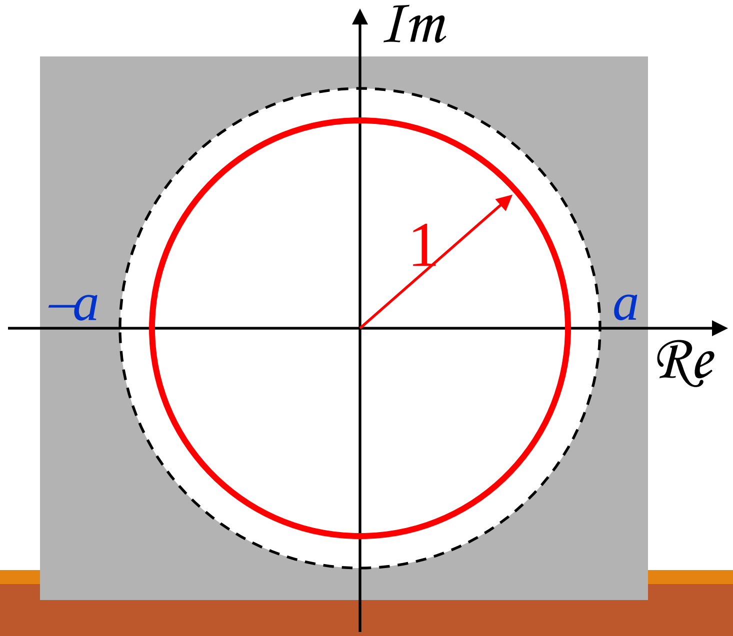
$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

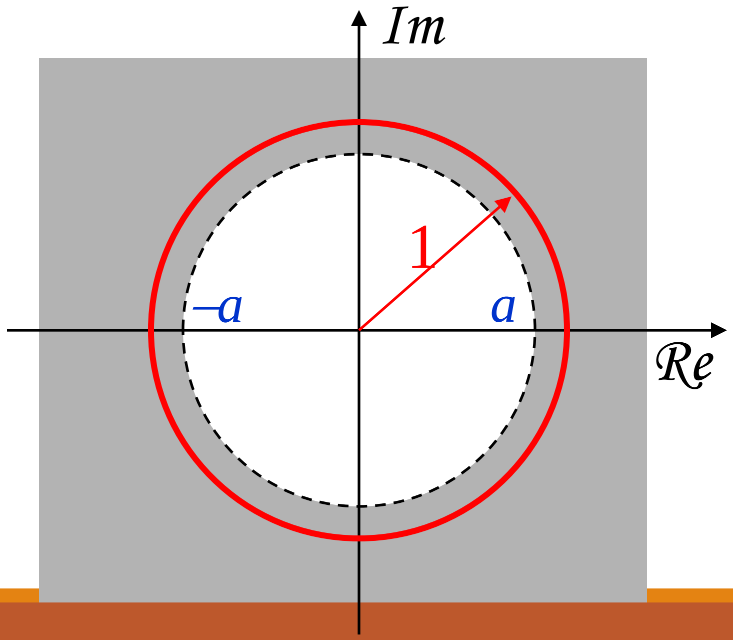
$$|z| > |a|$$

Example: A right sided Sequence ROC for $x(n)=a^n u(n)$

$$X(z) = \frac{z}{z-a}, \qquad |z| > |a|$$

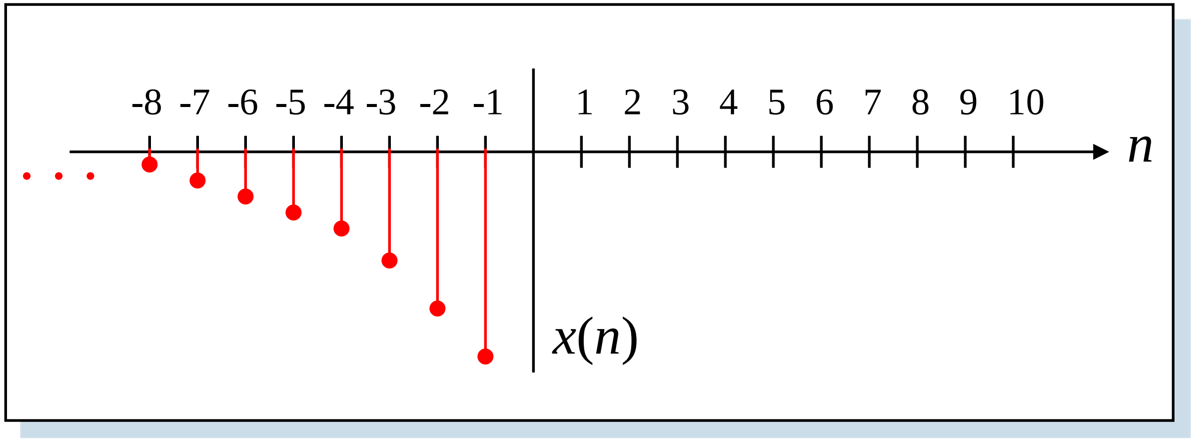
Which one is stable?





Example: A left sided Sequence

$$x(n) = -a^n u(-n-1)$$



Example: A left sided Sequence

$$\boxed{x(n) = -a^n u(-n-1)}$$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n$$

$$= 1 - \sum_{n=0}^{\infty} a^{-n} z^n$$

For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} | a^{-1} z | < \infty \quad \Rightarrow \quad | a^{-1} z | < 1$$

$$\Rightarrow \quad | z | < | a |$$

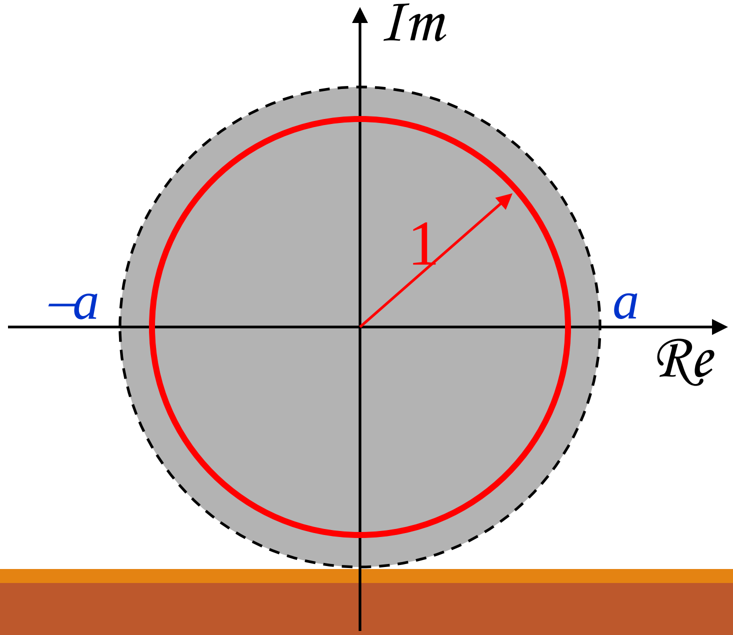
$$X(z) = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n = 1 - \frac{1}{1 - a^{-1} z} = \frac{z}{z - a}$$

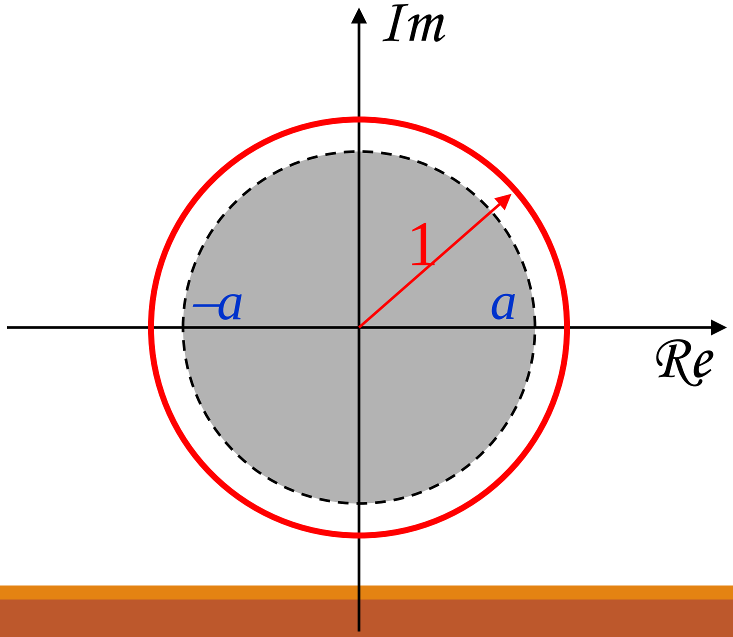
$$| z | < | a |$$

Example: A left sided Sequence ROC for $x(n) = -a^n u(-n-1)$

$$X(z) = \frac{z}{z-a}, \qquad |z| < |a|$$

Which one is stable?





REGION OF CONVERGENCE

The z-Transform

Represent *z*-transform as a Rational Function

$$X(z) = \frac{P(z)}{Q(z)}$$

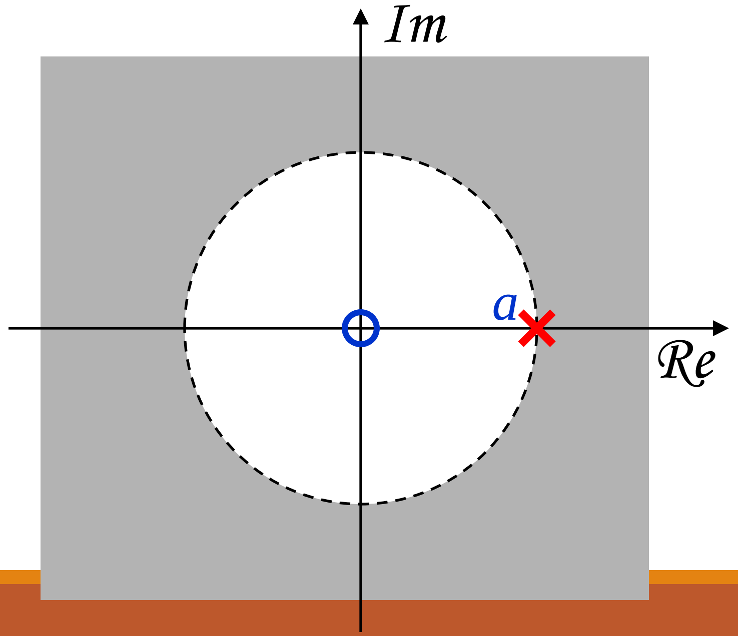
where $P(z)$ and $Q(z)$ are polynomials in z.

Zeros: The values of z’s such that $X(z) = 0$

Poles: The values of z’s such that $X(z) = \infty$

Example: A right sided Sequence

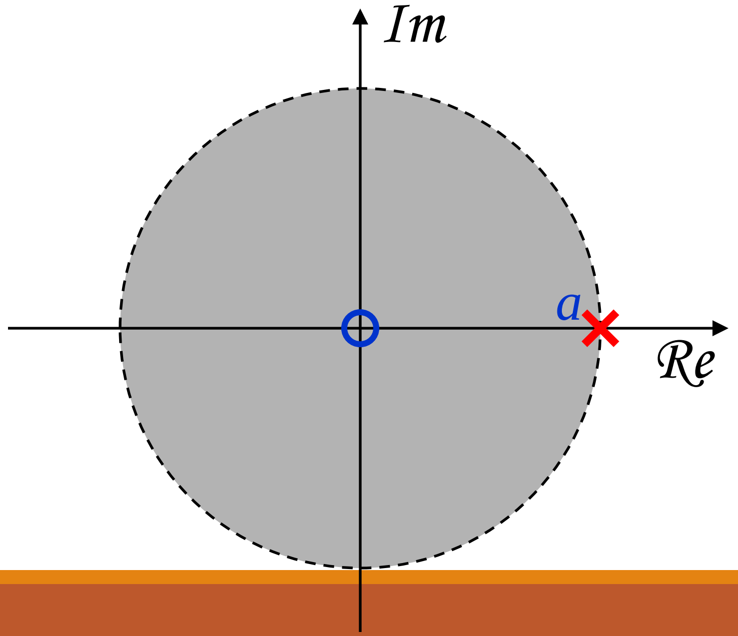
$$x(n) = a^n u(n) \Rightarrow X(z) = \frac{z}{z-a}, \qquad |z| > |a|$$



ROC is bounded by the pole and is the exterior of a circle.

Example: A left sided Sequence

$$x(n) = -a^n u(-n-1) \quad \Rightarrow \quad X(z) = \frac{z}{z-a}, \qquad |z| < |a|$$

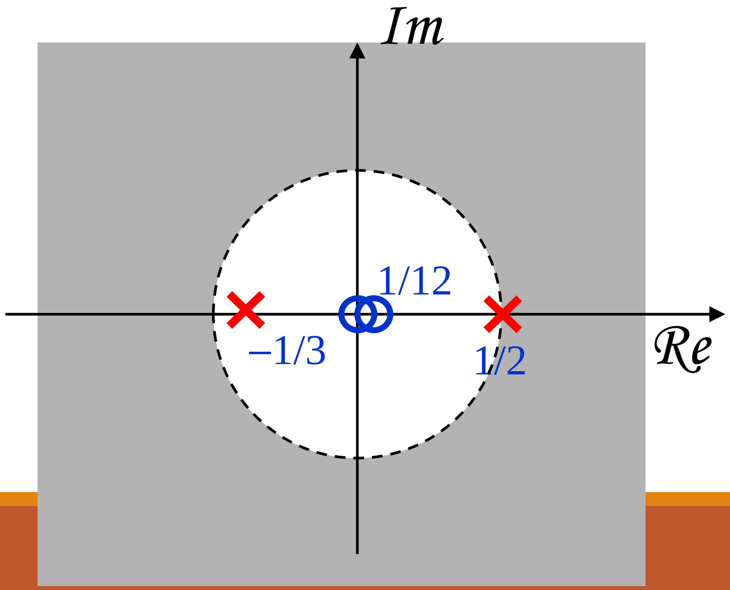


ROC is bounded by the pole and is the interior of a circle.

Example: Sum of Two Right Sided Sequences

$$x(n) = (\tfrac{1}{2})^n u(n) + (-\tfrac{1}{3})^n u(n)$$

$$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}} = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$$



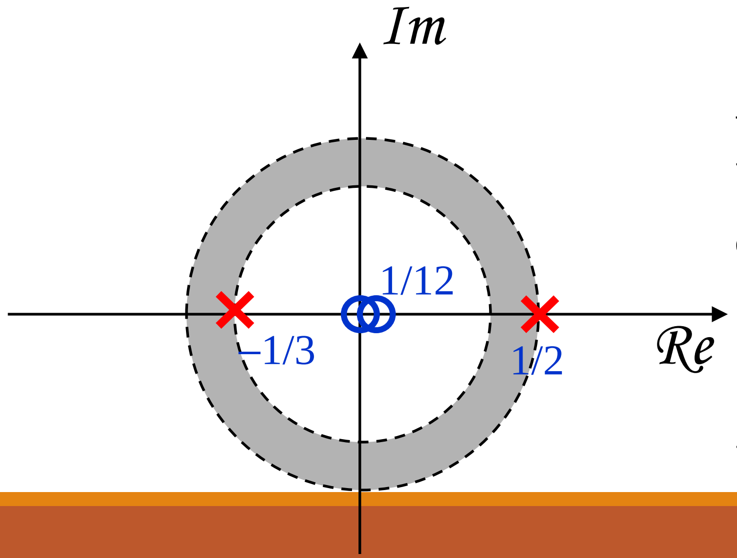
ROC is bounded by poles and is the exterior of a circle.

ROC does not include any pole.

Example: A Two Sided Sequence

$$x(n) = (-\tfrac{1}{3})^n u(n) - (\tfrac{1}{2})^n u(-n-1)$$

$$\Rightarrow \quad X(z) = \frac{z}{z+\frac{1}{3}} + \frac{z}{z-\frac{1}{2}} \quad = \frac{2z(z-\frac{1}{12})}{(z+\frac{1}{3})(z-\frac{1}{2})}$$



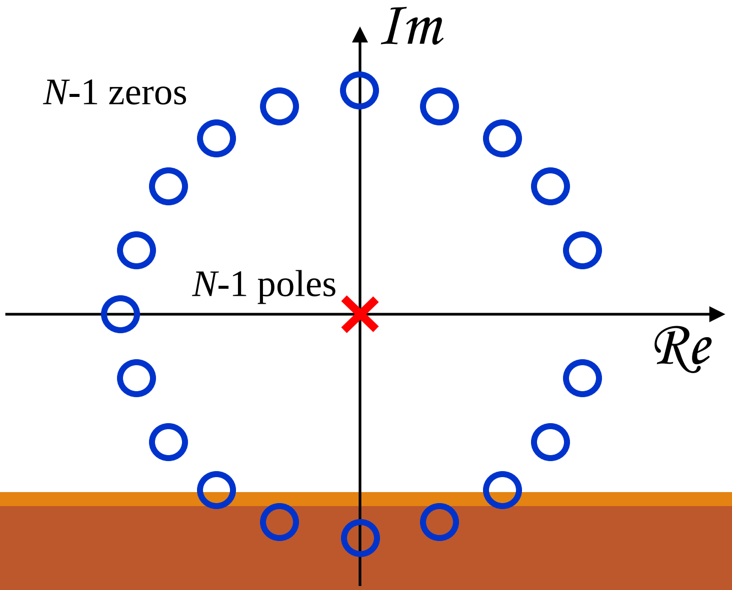
ROC is bounded by poles and is a ring.

ROC does not include any pole.

Example: A Finite Sequence

$$x(n) = a^n, \qquad 0 \le n \le N-1 \Rightarrow$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$



ROC: $0 < z < \infty$

ROC does not include any pole.

Always Stable

Properties of ROC

A ring or disk in the z-plane centered at the origin.

The Fourier Transform of $x(n)$ is converge absolutely iff the ROC includes the unit circle.

The ROC cannot include any poles

Finite Duration Sequences: The ROC is the entire z-plane except possibly $z=0$ or $z=\infty$.

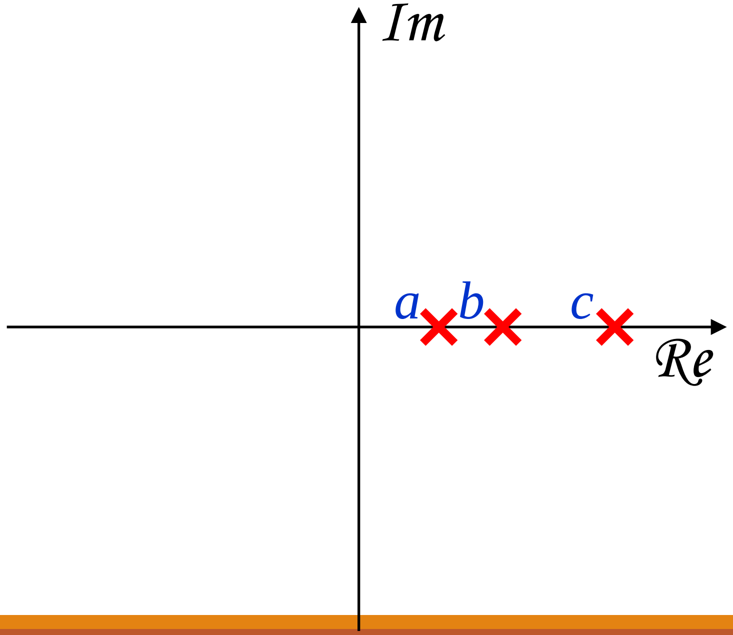
Right sided sequences: The ROC extends outward from the outermost finite pole in $X(z)$ to $z=\infty$.

Left sided sequences: The ROC extends inward from the innermost nonzero pole in $X(z)$ to $z=0$.

More on Rational *z*-Transform

Consider the rational *z*-transform with the pole pattern:

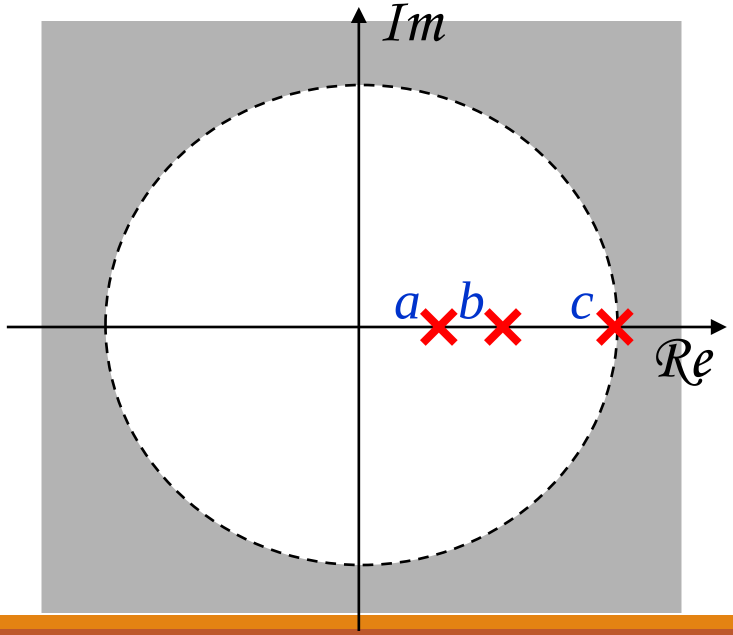
Find the possible ROC's



More on Rational *z*-Transform

Consider the rational *z*-transform with the pole pattern:

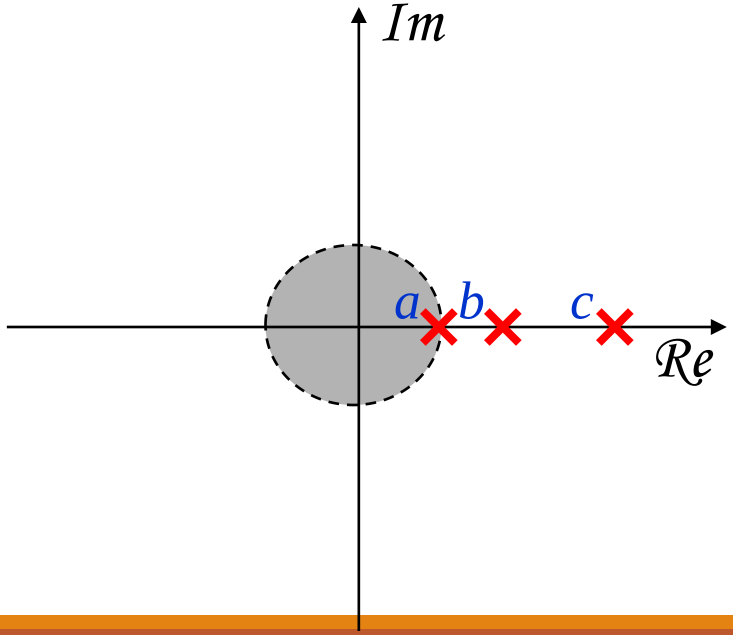
Case 1: A right sided Sequence.



More on Rational *z*-Transform

Consider the rational *z*-transform with the pole pattern:

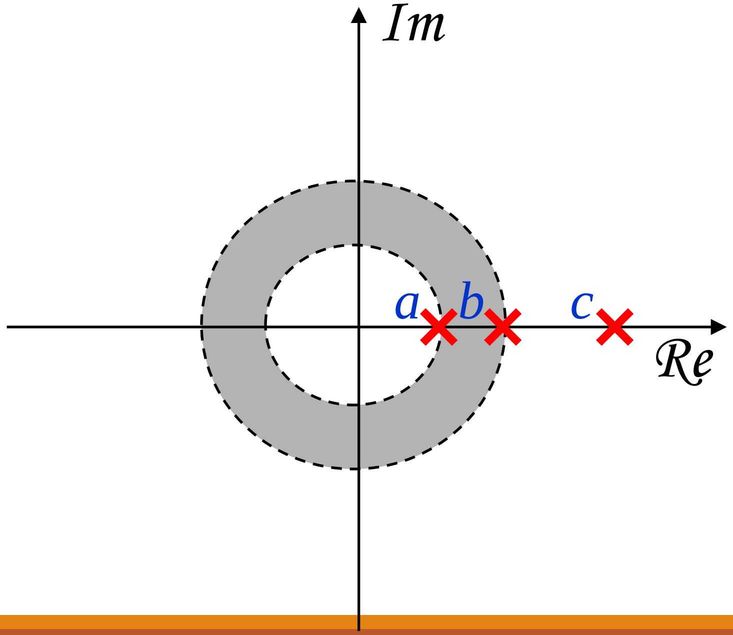
Case 2: A left sided Sequence.



More on Rational *z*-Transform

Consider the rational *z*-transform with the pole pattern:

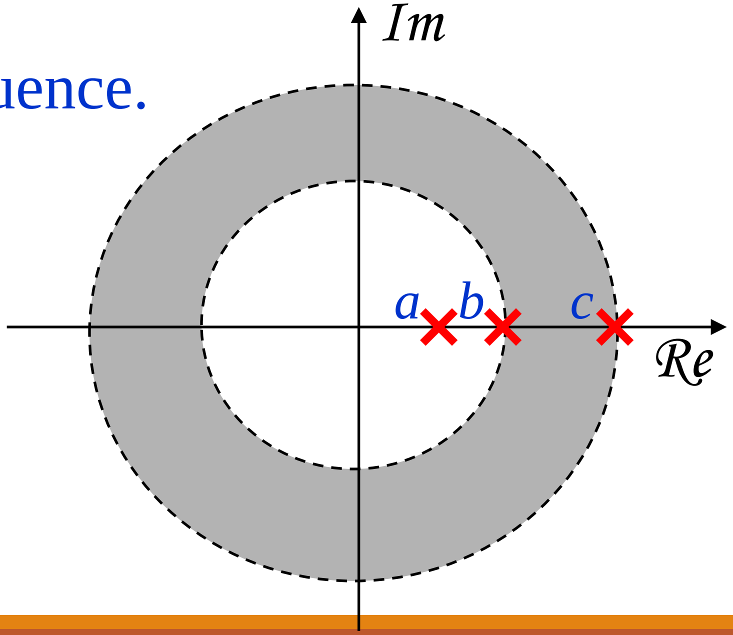
Case 3: A two sided Sequence.



More on Rational *z*-Transform

Consider the rational *z*-transform with the pole pattern:

Case 4: Another two sided Sequence.



IMPORTANT

Z-TRANSFORM PAIRS

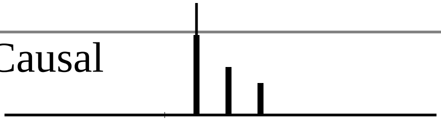
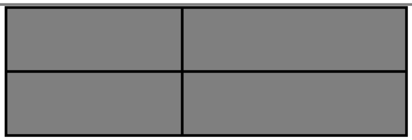
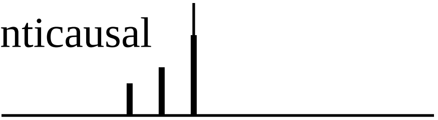
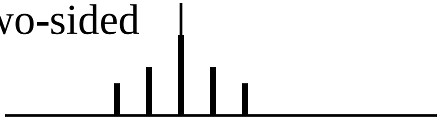
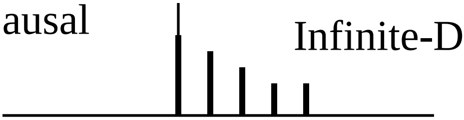
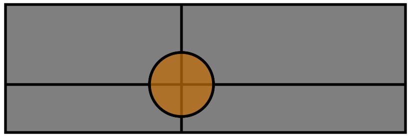
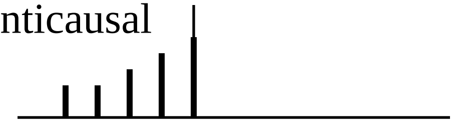
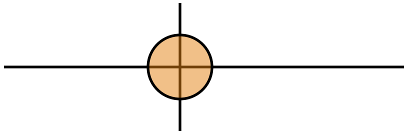
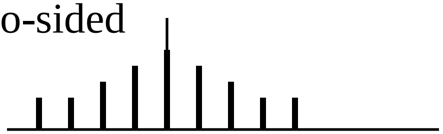
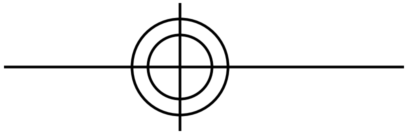
The z-Transform

Z-Transform Pairs

Sequence	*z*-Transform	ROC
$\delta(n)$	1	All *z*
$\delta(n-m)$	z^{-m}	All *z* except 0 (if *m*>0) or ∞ (if *m*<0)
$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u(-n-1)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$

Z-Transform Pairs

Sequence	z-Transform	ROC
$[\cos\omega_0 n]u(n)$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert>1$
$[\sin\omega_0 n]u(n)$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert>1$
$[r^n\cos\omega_0 n]u(n)$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert>r$
$[r^n\sin\omega_0 n]u(n)$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert>r$
$\begin{cases} a^n & 0\le n\le N-1 \\ 0 & \text{otherwise}\end{cases}$	$\dfrac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert>0$

Signal Type		ROC	
	Finite-Duration Signals		
Causal		Entire z-plane Except $z = 0$	
Anticausal		Entire z-plane Except $z =$ infinity	
Two-sided		Entire z-plane Except $z = 0$ And $z =$ infinity	
	Infinite-Duration Signals		
Causal		$\|z\| > r_2$	
Anticausal		$\|z\| < r_1$	
Two-sided		$r_2 < \|z\| < r_1$	

Some Common z-Transform Pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	all z
2. $u[n]$	$z/(z-1)$	$\|z\|>1$
3. $-u[-n-1]$	$z/(z-1)$	$\|z\|<1$
4. $\delta[n-m]$	z^{-m}	all z except 0 if $m>0$ or ฅ if $m<0$
5. $a^n u[n]$	$z/(z-a)$	$\|z\|>\|a\|$
6. $-a^n u[-n-1]$	$z/(z-a)$	$\|z\|<\|a\|$
7. $na^n u[n]$	$az/(z-a)^2$	$\|z\|>\|a\|$
8. $-na^n u[-n-1]$	$az/(z-a)^2$	$\|z\|<\|a\|$
9. $[\cos\omega_0 n]u[n]$	$(z^2-[\cos\omega_0]z)/(z^2-[2\cos\omega_0]z+1)$	$\|z\|>1$
10. $[\sin\omega_0 n]u[n]$	$[\sin\omega_0]z)/(z^2-[2\cos\omega_0]z+1)$	$\|z\|>1$
11. $[r^n\cos\omega_0 n]u[n]$	$(z^2-[r\cos\omega_0]z)/(z^2-[2r\cos\omega_0]z+r^2)$	$\|z\|>r$
12. $[r^n\sin\omega_0 n]u[n]$	$[r\sin\omega_0]z)/(z^2-[2r\cos\omega_0]z+r^2)$	$\|z\|>r$
13. $a^n u[n] - a^n u[n-N]$	$(z^N-a^N)/z^{N-1}(z-a)$	$\|z\|>0$

INVERSE Z-TRANSFORM

The z-Transform

Inverse Z-Transform by Partial Fraction Expansion

Assume that a given z-transform can be expressed as

$$X(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$X(z)=\sum_{r=0}^{M-N} B_r z^{-r}+\sum_{k=1,k\neq i}^{N}\frac{A_k}{1-d_k z^{-1}}+\sum_{m=1}^{s}\frac{C_m}{\left(1-d_i z^{-1}\right)^m}$$

Apply partial fractional expansion

First term exist only if M>N

- B_r is obtained by long division

Second term represents all first order poles

Third term represents an order s pole

- There will be a similar term for every high-order pole

Each term can be inverse transformed by inspection

Partial Fractional Expression

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{\left(1 - d_i z^{-1}\right)^m}$$

Coefficients are given as

$$A_k = \left(1 - d_k z^{-1}\right) X(z) \Big|_{z=d_k}$$

$$C_m = \frac{1}{(s-m)!\,(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} \left[\left(1 - d_i w\right)^s X\left(w^{-1}\right) \right] \right\}_{w=d_i^{-1}}$$

Easier to understand with examples

Example: 2nd Order Z-Transform

$$X(z) = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \qquad \text{ROC}: |z| > \frac{1}{2}$$

$$X(z) = \frac{A_1}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)}$$

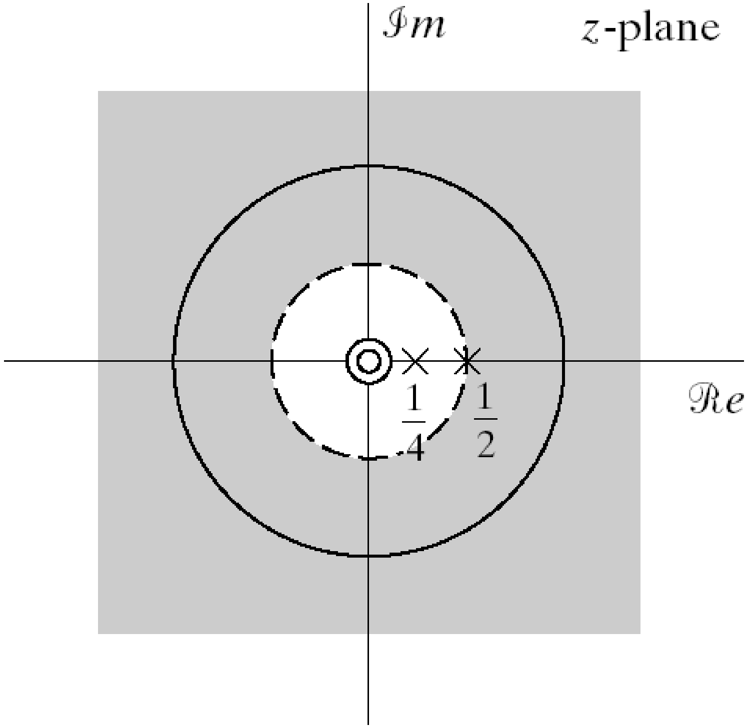
- Order of nominator is smaller than denominator (in terms of z^{-1})
- No higher order pole

$$A_1 = \left(1-\frac{1}{4}z^{-1}\right)X(z)\Big|_{z=\frac{1}{4}} = \frac{1}{\left(1-\frac{1}{2}\left(\frac{1}{4}\right)^{-1}\right)} = -1$$

$$A_2 = \left(1-\frac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}} = \frac{1}{\left(1-\frac{1}{4}\left(\frac{1}{2}\right)^{-1}\right)} = 2$$

Example Continued

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)} \qquad |z| >$$



ROC extends to infinity

- Indicates right sided sequence

$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

Example #2

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{\left(1 + z^{-1}\right)^2}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)} \qquad |z| > 1$$

Long division to obtain B_o

$$\begin{array}{r|l} & \hfill 2 \\ \cline{2-2} \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & \underline{z^{-2} - 3z^{-1} + 2} \\ & \hfill 5z^{-1} - 1 \end{array}$$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}$$

$$X(z) = 2 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

$$A_1 = \left(1 - \frac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}} = -9$$

$$A_2 = \left(1 - z^{-1}\right)X(z)\Big|_{z=1} = 8$$

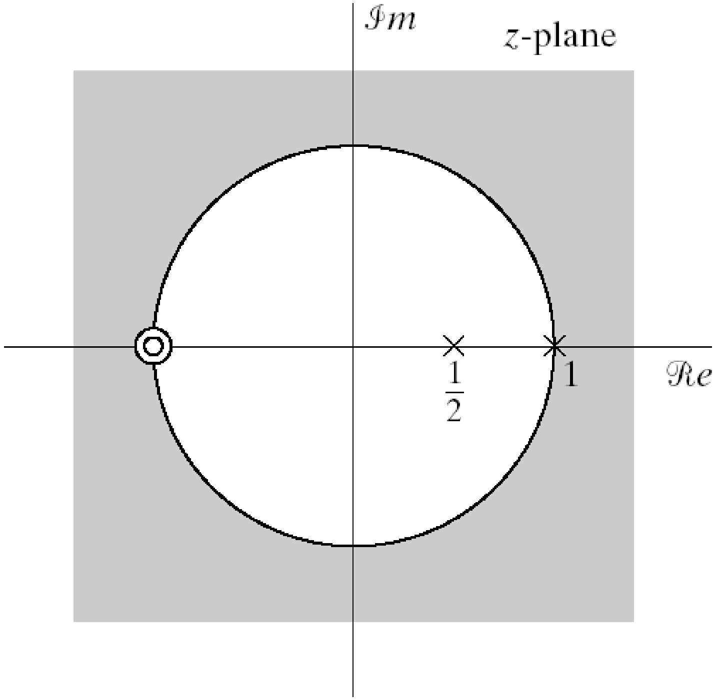
Example #2 Continued

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}} \qquad |z| > 1$$

ROC extends to infinity

- Indicates right-sides sequence

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] \text{ - } 8u[n]$$



An Example – Complete Solution

$$U(z) = \frac{3z^2 - 14z + 14}{z^2 - 6z + 8}$$

$$c_0 = \lim_{z \to \infty} U(z) = \lim_{z \to \infty} \frac{3z^2 - 14z + 14}{z^2 - 6z + 8} = 3$$

$$U(z) = c_0 + \frac{c_1}{z-2} + \frac{c_2}{z-4}$$

$$U_2(z) = (z-2)\frac{3z^2 - 14z + 14}{z^2 - 6z + 8}$$

$$= \frac{3z^2 - 14z + 14}{z - 4}$$

$$c_1 = U_2(2) = \frac{3 \cdot 2^2 - 14 \cdot 2 + 14}{2 - 4} = 1$$

$$U_4(z) = (z-4)\frac{3z^2 - 14z + 14}{z^2 - 6z + 8}$$

$$= \frac{3z^2 - 14z + 14}{z - 2}$$

$$c_2 = U_4(4) = \frac{3 \cdot 4^2 - 14 \cdot 4 + 14}{4 - 2} = 3$$

$$U(z) = 3 + \frac{1}{z-2} + \frac{3}{z-4}$$

$$u(k) = \begin{cases} 3, & k = 0 \\ 2^{k-1} + 3 \cdot 4^{k-1}, & k > 0 \end{cases}$$

Inverse Z-Transform by Power Series Expansion

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

The z-transform is power series

$$X(z) = \cdots + x[-2]z^{2} + x[-1]z^{1} + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

In expanded form

Z-transforms of this form can generally be inversed easily

$$X(z) = z^{2}\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right)$$

Especially useful for finite-length series

Example

$$= z^{2} - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

$$x[n] = \begin{cases} 1 & n = -2 \\ -\frac{1}{2} & n = -1 \\ -1 & n = 0 \\ \frac{1}{2} & n = 1 \\ 0 & n = 2 \end{cases}$$

Z-Transform Properties: Linearity

Notation $x[n] \xleftrightarrow{Z} X(z)$ $\mathrm{ROC} = R_x$

Linearity

$$ax_1[n] + bx_2[n] \xleftrightarrow{Z} aX_1(z) + bX_2(z) \qquad \mathrm{ROC} = R_{x_1} \cap R_{x_2}$$

$$x[n] = a^n u[n] - a^n u[n - N]$$

- Note that the ROC of combined sequence may be larger than either ROC
- This would happen if some pole/zero cancellation occurs
- Example:
  - Both sequences are right-sided
  - Both sequences have a pole z=a
  - Both have a ROC defined as |z|>|a|
  - In the combined sequence the pole at z=a cancels with a zero at z=a
  - The combined ROC is the entire z plane except z=0

We did make use of this property already, where?

Z-Transform Properties: Time Shifting

$$x[n-n_o] \xleftrightarrow{Z} z^{-n_o}X(z) \qquad \text{ROC} = R_x$$

Here n_o is an integer

- If positive the sequence is shifted right
- If negative the sequence is shifted left

The ROC can change the new term may

- Add or remove poles at z=0 or z=∞

Example

$$X(z) = z^{-1}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right) \qquad |z| > \frac{1}{4}$$

$$x[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1]$$

Z-Transform Properties: Multiplication by Exponential

C is scaled by $|z_o|$

$$z_o^n x[n] \xleftrightarrow{Z} X(z / z_o) \qquad ROC = |z_o| R_x$$

All pole/zero locations are scaled

If z_o is a positive real number: z-plane shrinks or expands

If z_o is a complex number with unit magnitude it rotates

Example: We know the z-transform pair

$$u[n] \xleftrightarrow{Z} \frac{1}{1 - z^{-1}} \qquad \text{ROC} : |z| > 1$$

$$x[n] = r^n \cos(\omega_o n) u[n] = \frac{1}{2}\left(re^{j\omega_o}\right)^n u[n] + \frac{1}{2}\left(re^{-j\omega_o}\right)^n u[n]$$

Let’s find the z-transform of

$$X(z) = \frac{1/2}{1 - re^{j\omega_o} z^{-1}} + \frac{1/2}{1 - re^{-j\omega_o} z^{-1}} \qquad |z| > r$$

Z-Transform Properties: Differentiation

$$nx[n] \xleftrightarrow{\;Z\;} -z\frac{dX(z)}{dz} \qquad \text{ROC} = R_x$$

Example: We want the inverse z-transform of

$$X(z) = \log\left(1 + az^{-1}\right) \qquad |z| > |a|$$

Let’s differentiate to obtain rational expression

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1 + az^{-1}} \Rightarrow -z\frac{dX(z)}{dz} = az^{-1}\frac{1}{1 + az^{-1}}$$

$$nx[n] = a(-a)^{n-1}u[n-1]$$

Making use of z-transform properties and ROC

$$x[n] = (-1)^{n-1}\frac{a^n}{n}u[n-1]$$

Z-Transform Properties: Conjugation

$$x^*[n] \overset{Z}{\longleftrightarrow} X^*\left(z^*\right) \qquad \text{ROC} = R_x$$

Example

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$X^*(z) = \left(\sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \right)^* = \sum_{n=-\infty}^{\infty} x^*[n]\, z^{n}$$

$$X^*\left(z^*\right) = \sum_{n=-\infty}^{\infty} x^*[n] \left(z^n\right)^* = \sum_{n=-\infty}^{\infty} x^*[n]\, z^{-n} = Z\{x^*[n]\}$$

Z-Transform Properties: Time Reversal

$$x[-n] \xleftrightarrow{Z} X(1/z) \qquad \text{ROC} = \frac{1}{R_x}$$

ROC is inverted

$$x[n] = a^{-n}u[-n]$$

Example:

$$a^n u[n]$$

$$X(z) = \frac{1}{1 - az} = \frac{- a^{-1}z^{-1}}{1 - a^{-1}z^{-1}} \qquad |z| < |a^{-1}|$$

Time reversed version of

Z-Transform Properties: Convolution

$$x_1[n] * x_2[n] \xleftrightarrow{Z} X_1(z)X_2(z) \qquad \text{ROC} : R_{x1} \cap R_{x_2}$$

Convolution in time domain is multiplication in z-domain

Example:Let’s calculate the convolution of

$$x_1[n] = a^n u[n] \text{ and } x_2[n] = u[n]$$

$$X_1(z) = \frac{1}{1 - az^{-1}} \quad \text{ROC} : |z| > |a| \qquad X_2(z) = \frac{1}{1 - z^{-1}} \quad \text{ROC} : |z| > 1$$

Multiplications of z-transforms is

$$Y(z) = X_1(z)X_2(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})}$$

ROC: if |a|<1 ROC is |z|>1 if |a|>1 ROC is |z|>|a|

$$y[n] = \frac{1}{1 - a}\left(u[n] - a^{n+1}u[n]\right)$$

Partial fractional expansion of Y(z)

$$Y(z) = \frac{1}{1 - a}\left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - az^{-1}}\right) \qquad \text{asume ROC} : |z| > 1$$

Z-TRANSFORM THEOREMS AND PROPERTIES

The z-Transform

Linearity

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\mathcal{Z}[y(n)] = Y(z), \qquad z \in R_y$$

$$\Downarrow$$

$$\mathcal{Z}[ax(n) + by(n)] = aX(z) + bY(z), \qquad \underbrace{z \in R_x \cap R_y}_{\text{Overlay of the above two ROC's}}$$

Shift

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

⇓

$$\mathcal{Z}[x(n+n_0)] = z^{n_0} X(z) \qquad z \in R_x$$

Multiplication by an Exponential Sequence

$$\mathcal{Z}[x(n)] = X(z), \qquad R_{x-} < |z| < R_{x+}$$

$$\mathcal{Z}[a^n x(n)] = X(a^{-1}z) \qquad z \in |a| \cdot R_x$$

Differentiation of *X*(*z*)

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

⇓

$$\mathcal{Z}[nx(n)] = -z\frac{dX(z)}{dz} \qquad z \in R_x$$

Conjugation

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\Downarrow$$

$$\mathcal{Z}[x^*(n)] = X^*(z^*) \qquad z \in R_x$$

Reversal

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\mathcal{Z}[x(-n)] = X(z^{-1}) \qquad z \in 1/R_x$$

Real and Imaginary Parts

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\mathscr{R}e[x(n)] = \tfrac{1}{2}[X(z) + X^*(z^*)] \qquad z \in R_x$$

$$Im[x(n)] = \tfrac{1}{2j}[X(z) - X^*(z^*)] \qquad z \in R_x$$

Initial Value Theorem

$$x(n) = 0, \qquad \text{for } n < 0$$

$$\Downarrow$$

$$x(0) = \lim_{z \to \infty} X(z)$$

Convolution of Sequences

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\mathcal{Z}[y(n)] = Y(z), \qquad z \in R_y$$

$$\Downarrow$$

$$\mathcal{Z}[x(n) * y(n)] = X(z)Y(z) \qquad z \in R_x \cap R_y$$

Convolution of Sequences

$$x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k)$$

$$\mathcal{Z}[x(n) * y(n)] = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x(k)y(n-k) \right) z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k)z^{-n} \quad = \sum_{k=-\infty}^{\infty} x(k)z^{-k} \sum_{n=-\infty}^{\infty} y(n)z^{-n}$$

$$= X(z)Y(z)$$

SYSTEM FUNCTION

The z-Transform

Signal Characteristics from Z-Transform

If U(z) is a rational function, and

$$y(k) = a_1 y(k-1) + \ldots + a_n y(k-n) + b_1 u(k-1) + \ldots + b_m u(k-m)$$

Then Y(z) is a rational function, too

zeros

$$Y(z) = \frac{N(z)}{D(z)} = \frac{\prod_{i=1}^{n} (z - z_i)}{\prod_{j=1}^{m} (z - p_j)}$$

poles

Poles are more important – determine key characteristics of y(k)

Why are poles important?

Z domain

$$Y(z)=\frac{N(z)}{D(z)}=\frac{\prod_{i=1}^{n}(z-z_i)}{\prod_{j=1}^{m}(z-p_j)}=c_0+\sum_{j=1}^{m}\frac{c_j}{z-p_j}$$

poles

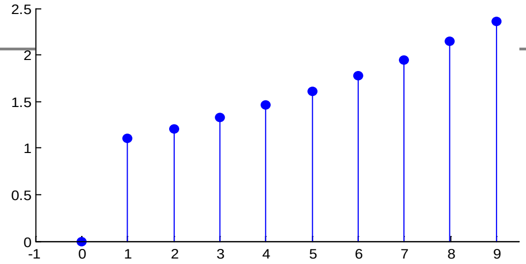
$\mathcal{Z}^{-1}$

Time domain

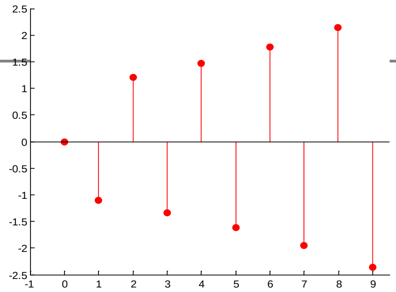
$$Y(k)=c_0\times u_{impulse}(k)+\sum_{j=1}^{m}c_j\times p_j^{k-1}$$

components

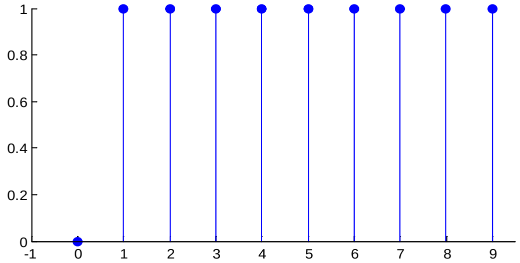
Various pole values (1)



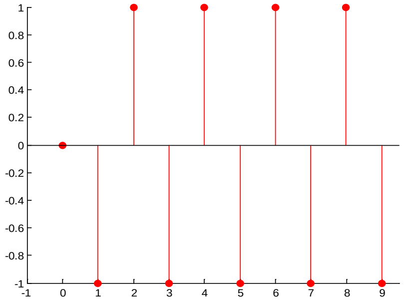
p=1.1



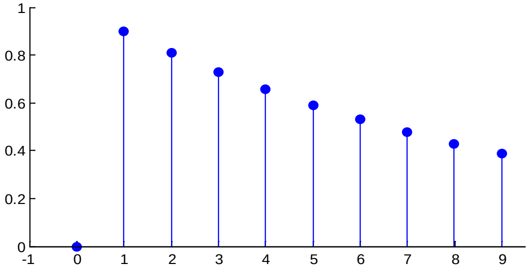
p=-1.1



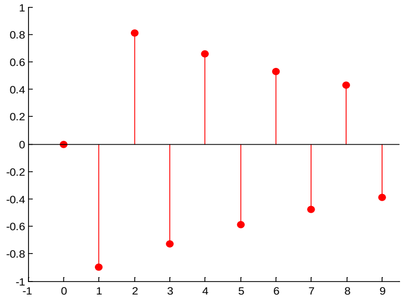
p=1



p=-1

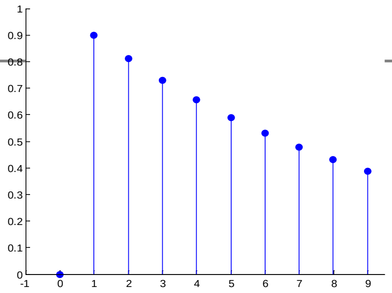


p=0.9

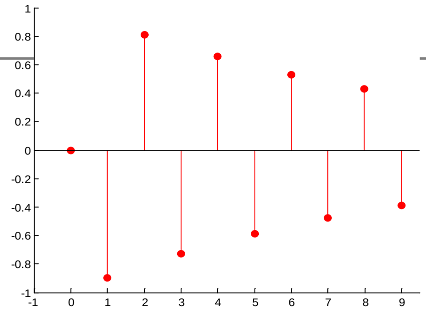


p=-0.9

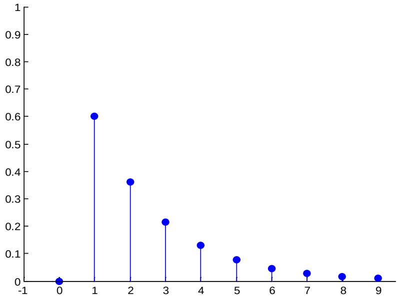
Various pole values (2)



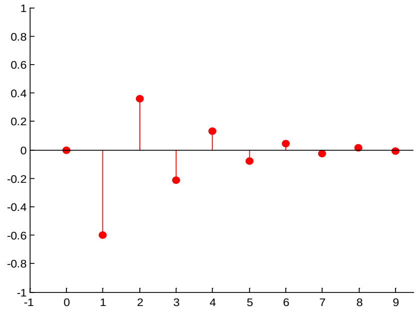
p=0.9



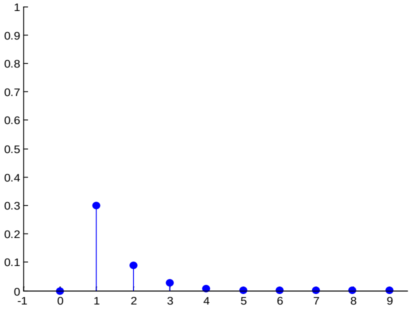
p=-0.9



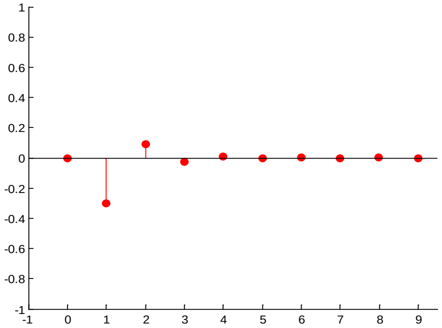
p=0.6



p=-0.6



p=0.3



p=-0.3

Conclusion for Real Poles

If and only if all poles’ absolute values are smaller than 1, y(k) converges to 0

The smaller the poles are, the faster the corresponding component in y(k) converges

A negative pole’s corresponding component is oscillating, while a positive pole’s corresponding component is monotonous

How fast does it converge?

U(k)=a^k, consider u(k)≈0 when the absolute value of u(k) is smaller than or equal to 2% of u(0)’s absolute value

$$|a|^k = 0.02$$

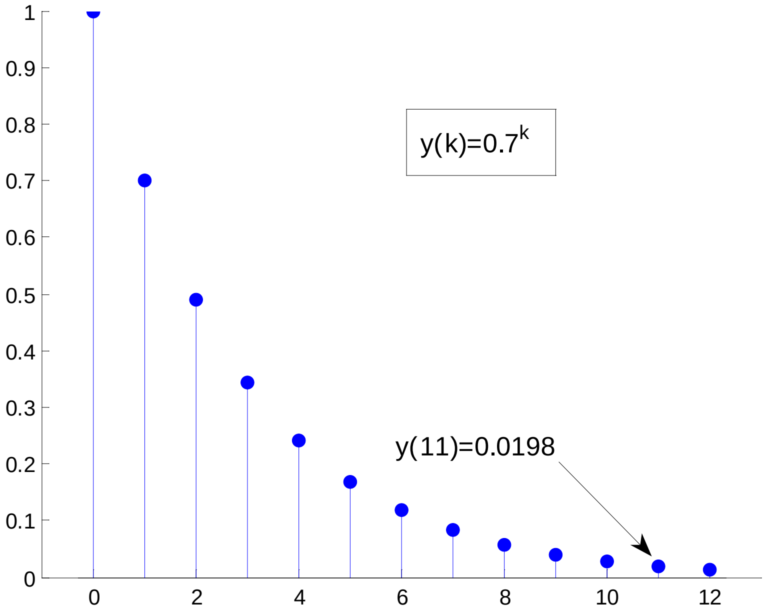
$$k\ln|a| = \ln 0.02 = -3.912$$

$$k \approx \frac{-4}{\ln|a|}$$

Remember This!

$$a = 0.7$$

$$k \approx \frac{-4}{\ln|0.7|} \approx \frac{-4}{-0.36} \approx 11$$



When There Are Complex Poles …

$$Y(z)=\frac{b_1z^{-1}+\ldots+b_mz^{-m}}{1-a_1z^{-1}-\ldots-a_nz^{-n}}U(z)$$

$$\longrightarrow (az^2+bz+c)\ldots$$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

If $b^2-4ac\geq 0,$

$$az^2+bz+c=a(z-\frac{-b+\sqrt{b^2-4ac}}{2a})(z-\frac{-b-\sqrt{b^2-4ac}}{2a})$$

If $b^2-4ac<0,$

$$az^2+bz+c=a(z-\frac{-b+i\sqrt{4ac-b^2}}{2a})(z-\frac{-b-i\sqrt{4ac-b^2}}{2a})$$

Or in polar coordinates,

$$az^2+bz+c=a(z-r\cos\theta-ir\sin\theta)(z-r\cos\theta+ir\sin\theta)$$

What If Poles Are Complex

If Y(z)=N(z)/D(z), and coefficients of both D(z) and N(z) are all real numbers, if p is a pole, then p’s complex conjugate must also be a pole

- Complex poles appear in pairs

$$Y(z)=c_0+\sum_{j=1}^{l}\frac{c_j}{z-p_j}+\frac{c}{z-r\cos\theta-ir\sin\theta}+\frac{c'}{z-r\cos\theta+ir\sin\theta}$$

$$=c_0+\sum_{j=1}^{l}\frac{c_j}{z-p_j}+\frac{bzr\sin\theta+dz(z-r\cos\theta)}{z^2-(2r\cos\theta)z+r^2}$$

$$\Big\downarrow\ \mathcal{Z}^{-1}$$

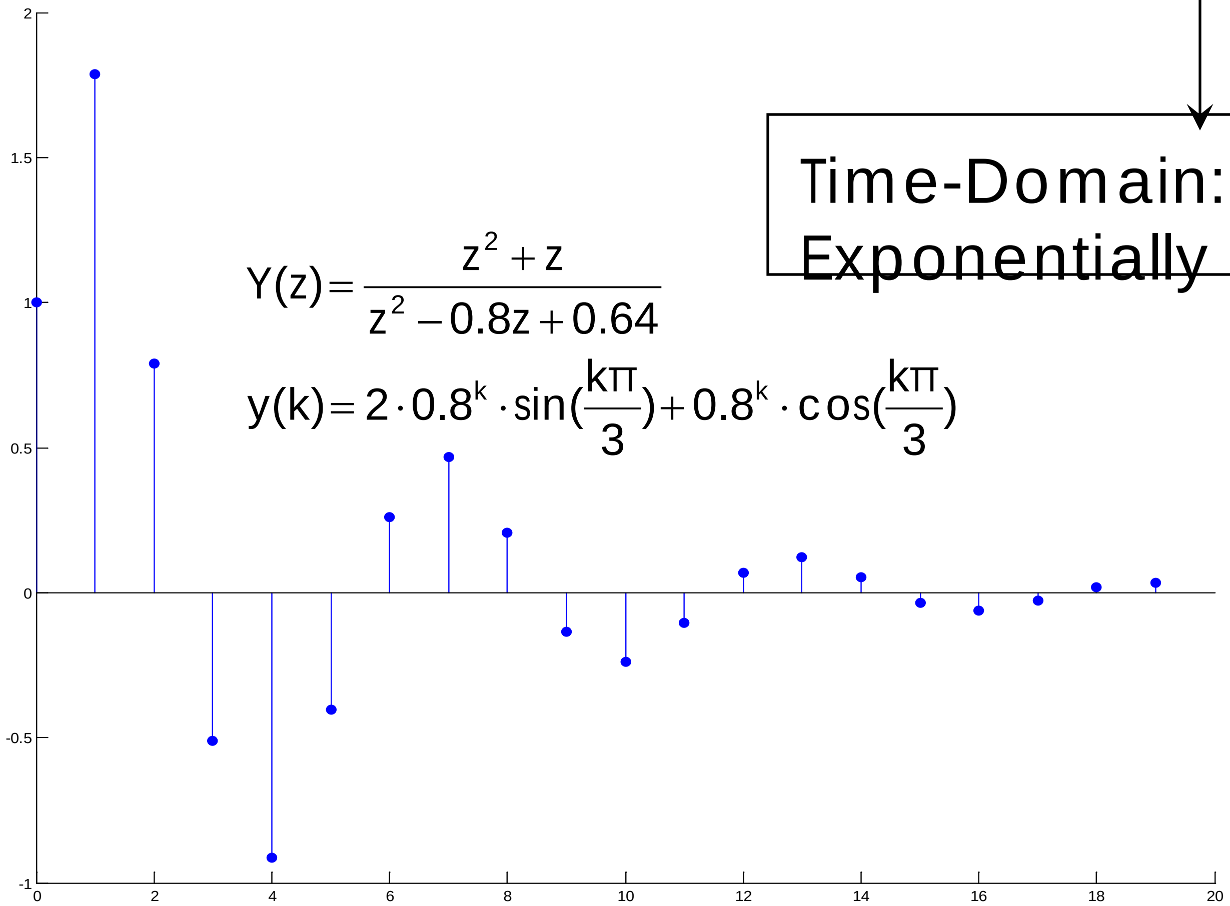
Time domain

$$y(k)=c_0\times u_{impulse}(k)+\sum_{j=1}^{m}c_j\times p_j^{k-1}+br^k\sin k\Theta+dr^k\cos k\Theta$$

An Example

Z-Domain: Complex Poles

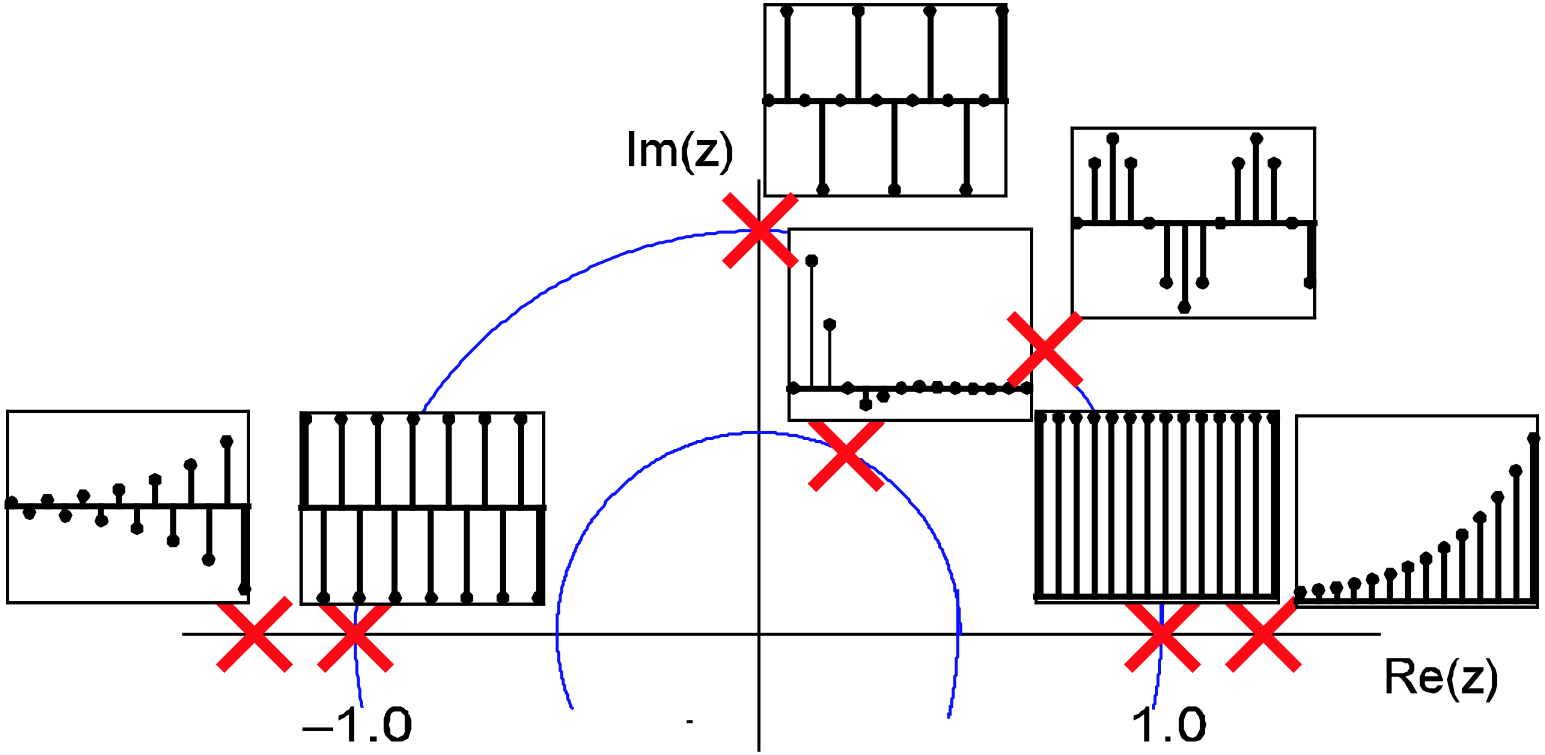
Time-Domain:
Exponentially Modulated Sin/



$$Y(z)=\frac{z^2+z}{z^2-0.8z+0.64}$$

$$y(k)=2\cdot 0.8^k\cdot \sin(\frac{k\pi}{3})+0.8^k\cdot \cos(\frac{k\pi}{3})$$

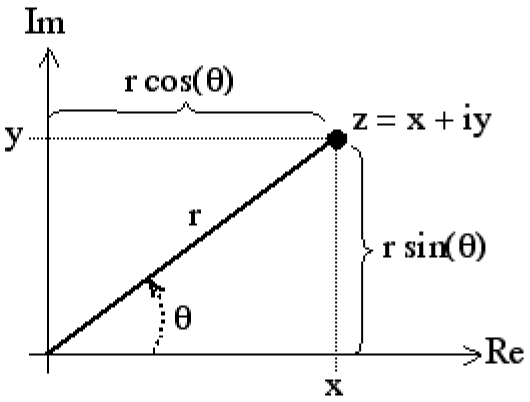
Poles Everywhere



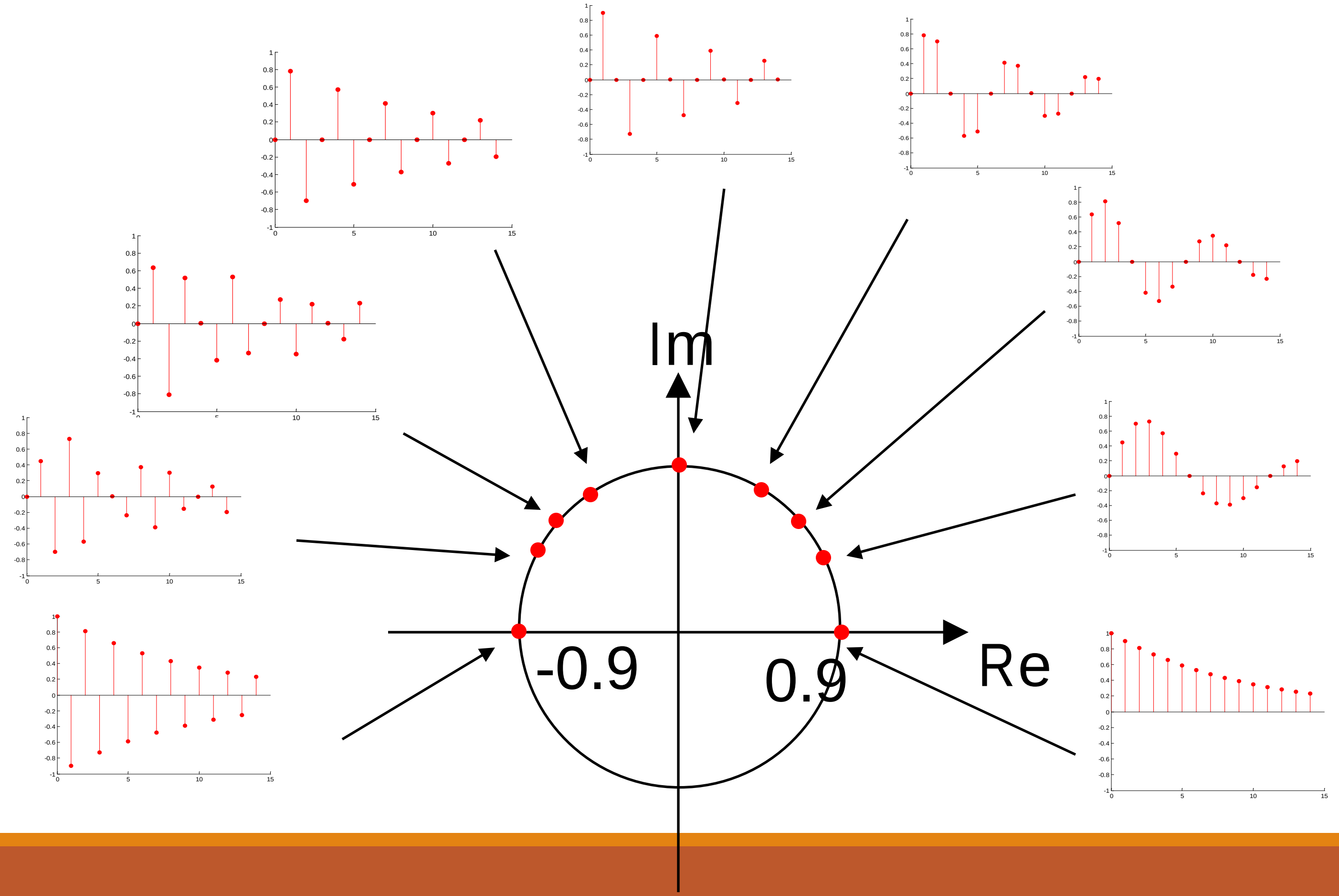
Observations

Using poles to characterize a signal

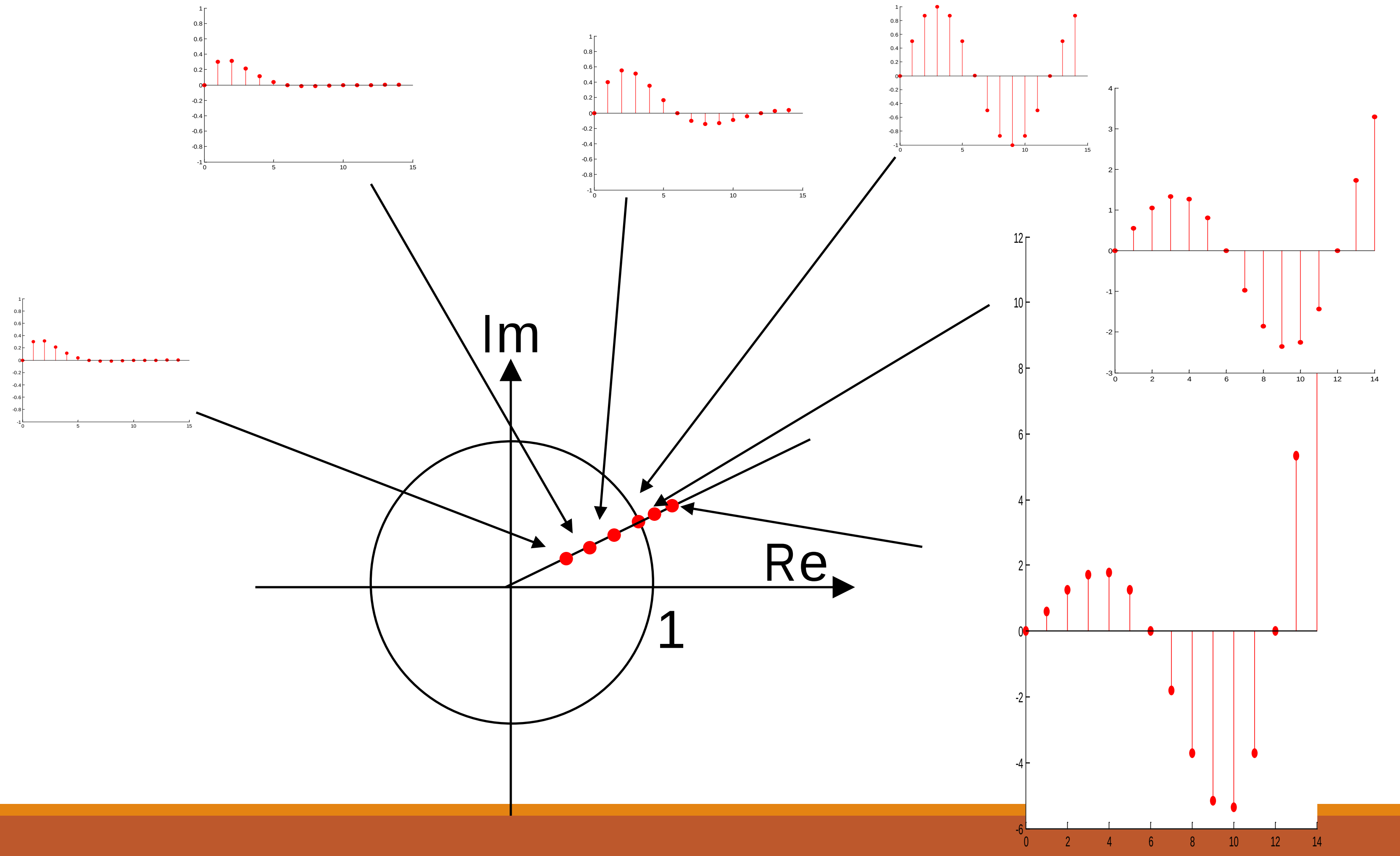
- The smaller is |r|, the faster converges the signal
  - |r| < 1, converge
  - |r| > 1, does not converge, unbounded
  - |r|=1?
- When the angle increase from 0 to pi, the frequency of oscillation increases
  - Extremes – 0, does not oscillate, pi, oscillate at the maximum frequency



Change Angles



Changing Absolute Value



Conclusion for Complex Poles

A complex pole appears in pair with its complex conjugate

The Z^{-1}-transform generates a combination of exponentially modulated sin and cos terms

The exponential base is the absolute value of the complex pole

The frequency of the sinusoid is the angle of the complex pole (divided by 2π)

Steady-State Analysis

If a signal finally converges, what value does it converge to?

When it does not converge

- Any $|p_j|$ is greater than 1
- Any $|r|$ is greater than or equal to 1

When it does converge

- If all $|p_j|$'s and $|r|$'s are smaller than 1, it converges to 0
- If only one p_j is 1, then the signal converges to c_j
  - If more than one real pole is 1, the signal does not converge … (e.g. the ramp signal)

$$y(k) = c_0 \times u_{impulse}(k) + \sum_{j=1}^{m} c_j \times p_j^{k-1} + br^k \sin k\theta + dr^k \cos k\theta$$

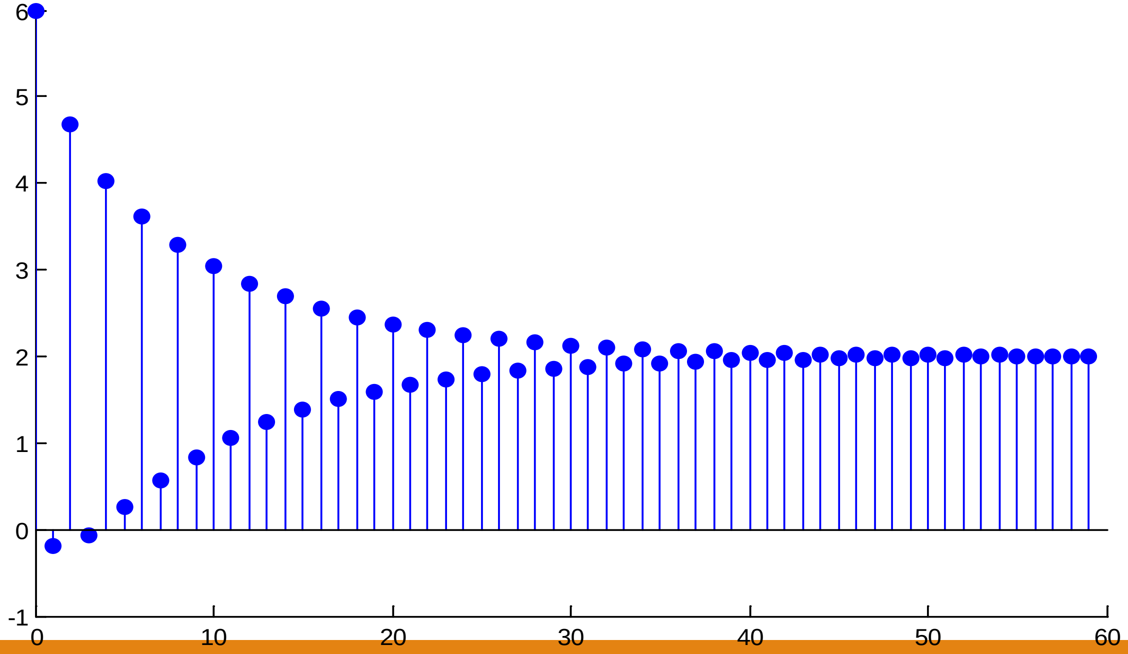
$$\frac{z^{-1}}{(1-z^{-1})^2}$$

An Example

$$U(z) = \frac{2z}{z-1} + \frac{z}{z-0.5} + \frac{3z}{z+0.9}$$

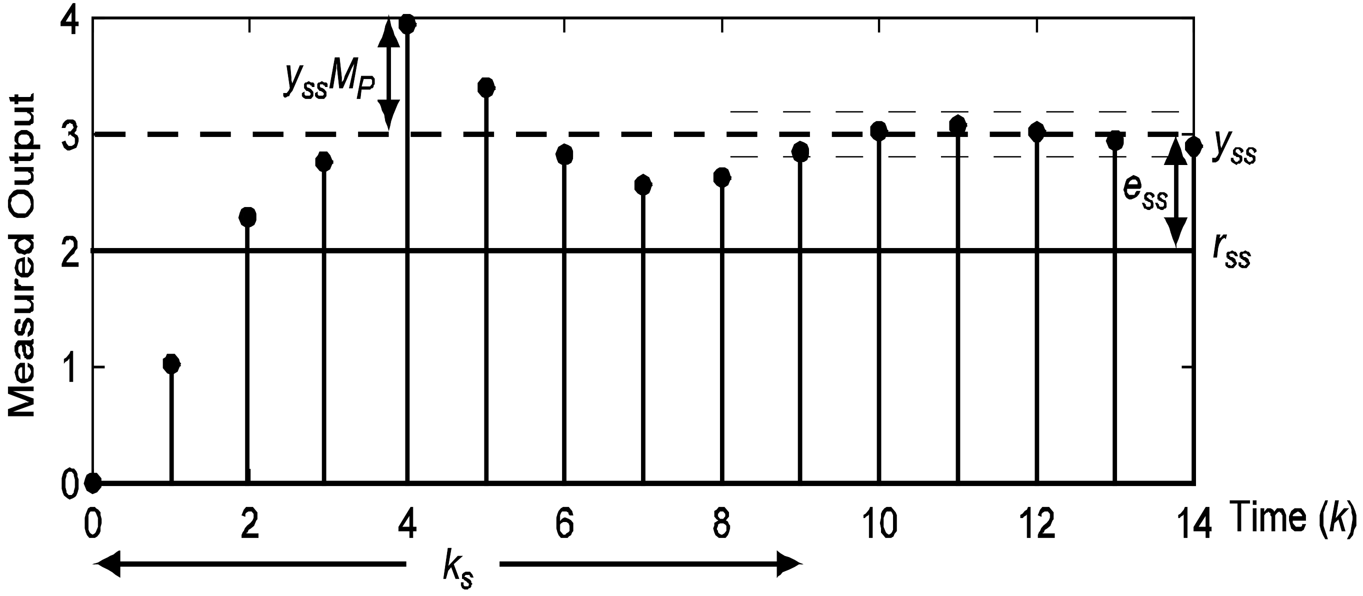
$$u(k) = 2 + 0.5^k + 3 \cdot (-0.9)^k$$

converge to 2



Final Value Theorem

Enable us to decide whether a system has a steady state error (y_{ss}-r_{ss})



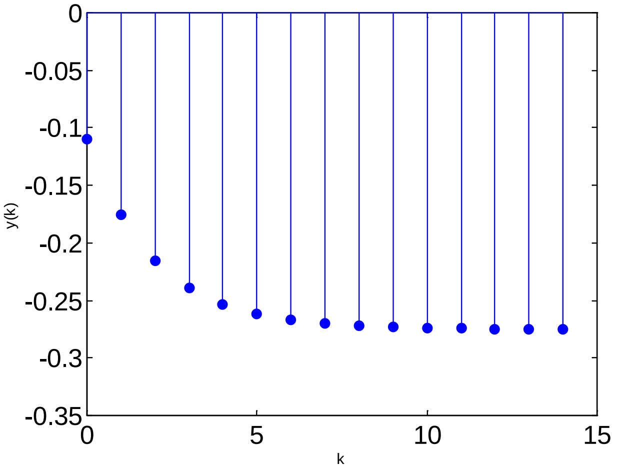
Final Value Theorem

Theorem: If all of the poles of $(1-z)Y(z)$ lie within the unit circle, then

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(z-1)Y(z)$$

$$Y(z) = \frac{-0.11z}{z^2 - 1.6z + 0.6} = \frac{-0.11z}{(z-1)(z-0.6)}$$

$$(z-1)Y(z)\,|_{z=1} = \frac{-0.11z}{z-0.6}\,|_{z=1} = -0.275$$

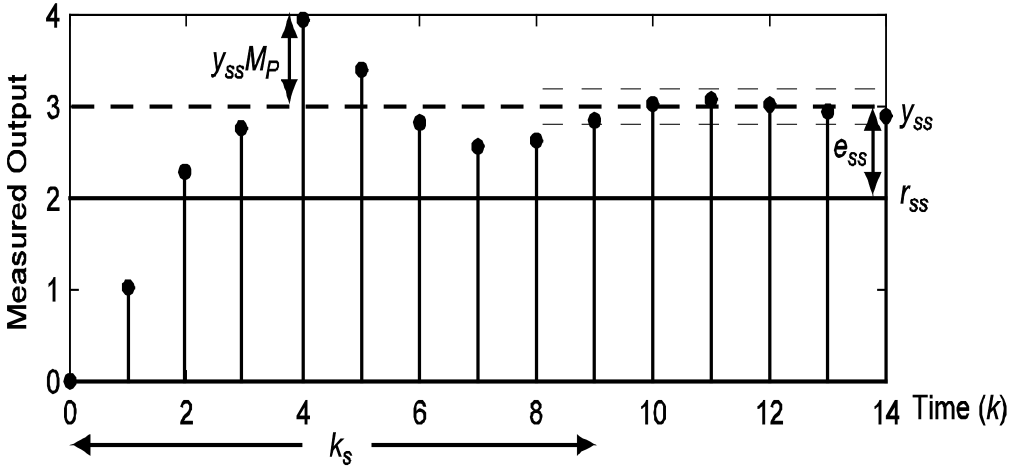


If any pole of (1-z)Y(z) lies out of or ON the unit circle, y(k) does not converge!

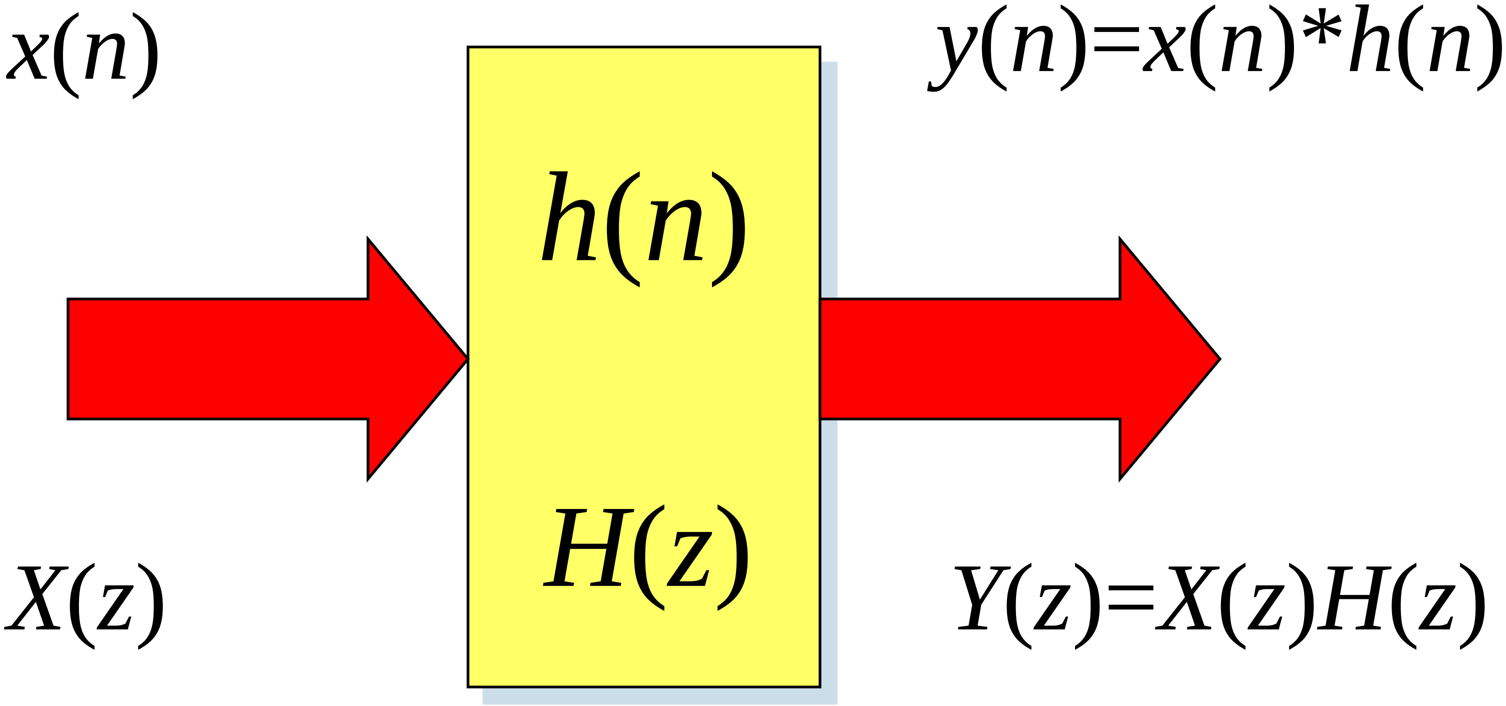
What Can We Infer from TF?

Almost everything we want to know

- Stability
- Steady-State
- Transients
  - Settling time
  - Overshoot
- …



Shift-Invariant System



$x(n)$

$y(n)=x(n)*h(n)$

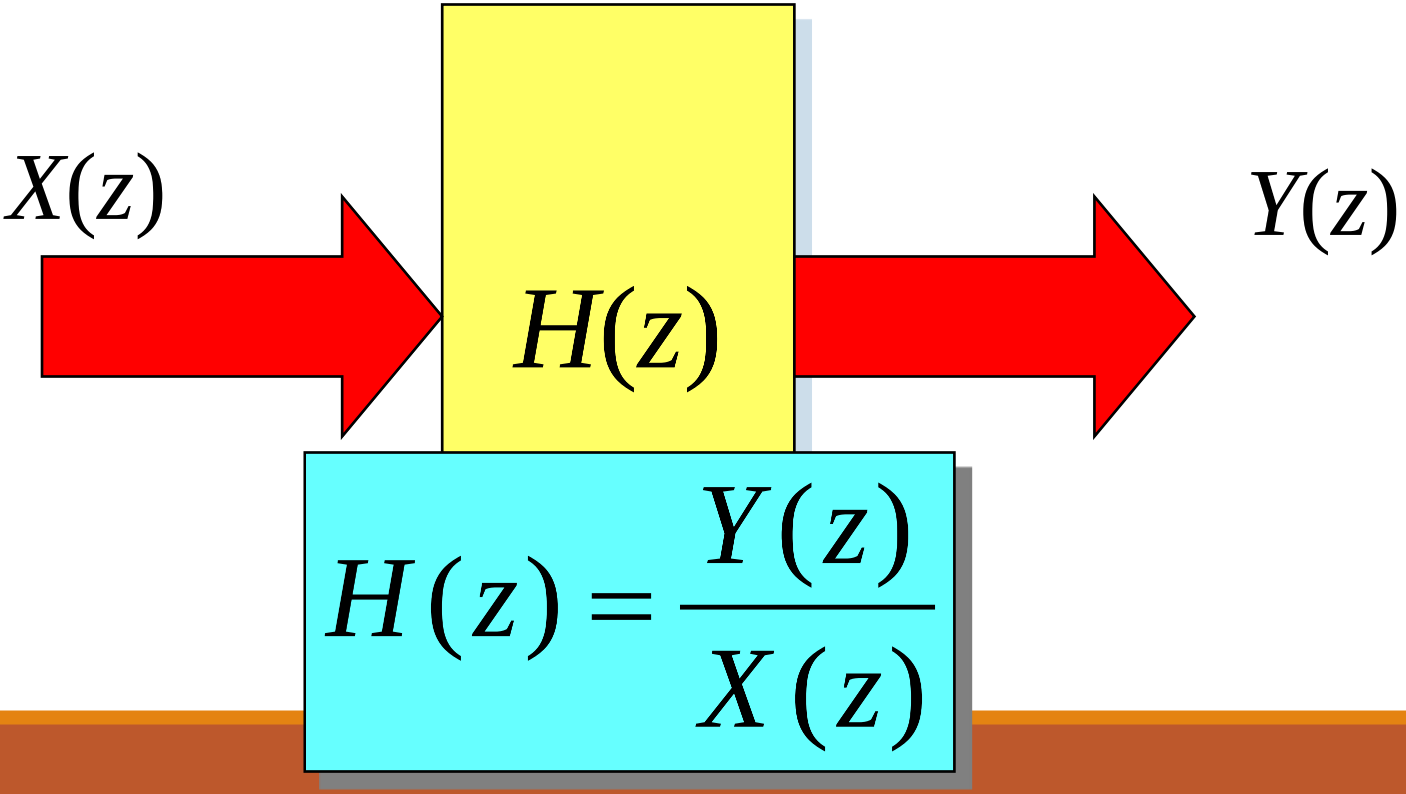
$h(n)$

$H(z)$

$X(z)$

$Y(z)=X(z)H(z)$

Shift-Invariant System



$$H(z) = \frac{Y(z)}{X(z)}$$

N^{th}-Order Difference Equation

$$\boxed{\sum_{k=0}^{N} a_k y(n-k) = \sum_{r=0}^{M} b_r x(n-r)}$$

➡

$$Y(z)\sum_{k=0}^{N} a_k z^{-k} = X(z)\sum_{r=0}^{M} b_r z^{-r}$$

$$\boxed{H(z) = \sum_{r=0}^{M} b_r z^{-r} \Big/ \sum_{k=0}^{N} a_k z^{-k}}$$

Representation in Factored Form

Contributes *poles* at 0 and *zeros* at c_r

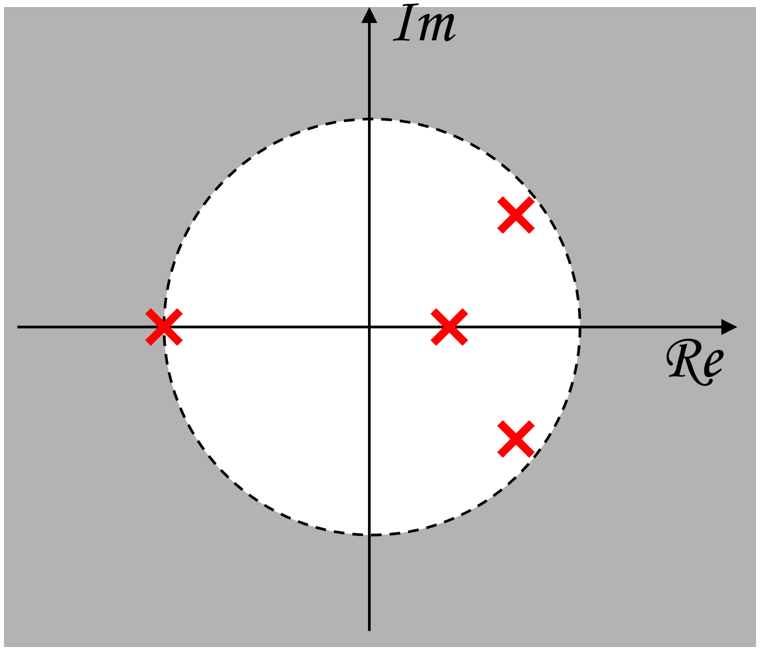
$$H(z) = \frac{A\prod_{r=1}^{M}(1-c_r z^{-1})}{\prod_{k=1}^{N}(1-d_r z^{-1})}$$

Contributes *zeros* at 0 and *poles* at d_r

Stable and Causal Systems

Causal Systems : ROC extends outward from the outermost pole.

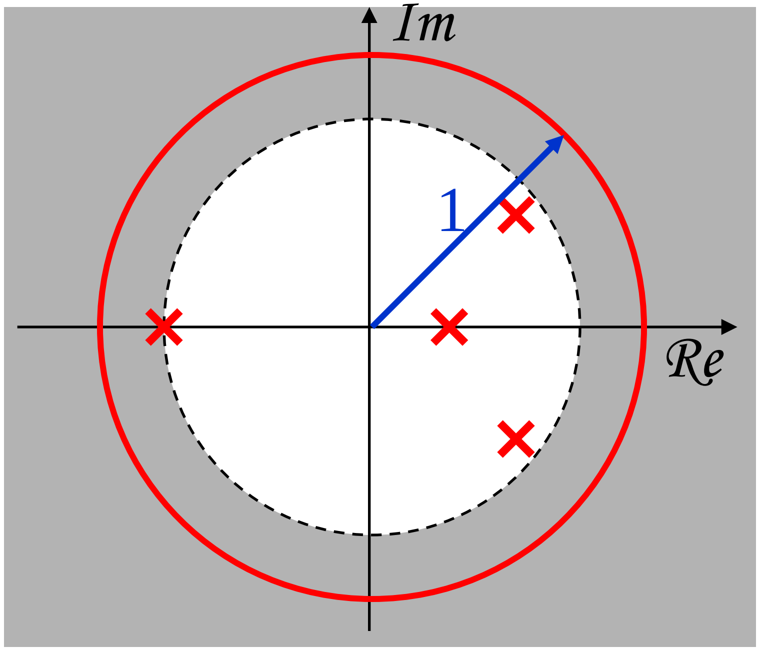
$$H(z)=\frac{A\prod_{r=1}^{M}(1-c_r z^{-1})}{\prod_{k=1}^{N}(1-d_r z^{-1})}$$



Stable and Causal Systems

Stable Systems : ROC includes the unit circle.

$$H(z)=\frac{A\prod_{r=1}^{M}(1-c_r z^{-1})}{\prod_{k=1}^{N}(1-d_r z^{-1})}$$



Thanks!

Reference :

https://akademik.adu.edu.tr/fakulte/muhendislik/personel/uploads/yilmaz.kalkan/dsp3-1577359228.ppt